Quantum mechanics II, Solutions 4 : Density operators

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Problem 1 : No signalling

Use your new-found understanding of reduced states to justify the no signalling principle (i.e. to argue why it is not possible to use entanglement to signal faster than light).

Solution

Intuitively

We can immediately intuit why signalling is impossible using our understanding of the reduced density matrix, which we know it is a description of a partition totally independent of the other partition(s). No matter what is performed upon the other partitions, the reduced density matrix is unchanged. Because the statistics of local measurements are informed entirely by expected values of operators upon the reduced density matrix, they are also independent of operations on other partitions. Ergo, signalling is impossible.

Two qubits

We can also prove this using density matrices, which we will find to be a much more pleasant process than our previous proofs using only statevectors. For simplicity, let's first consider Alice and Bob each have a qubit in a general, two-qubit pure state. This includes all possible entangled state. Alice proposes to perform an arbitrary measurement upon her qubit, collapsing it into either outcome state $|\lambda_1\rangle$ or $|\lambda_2\rangle$. We can use these states as a basis to express the general, pre-measurement state as

$$|\psi\rangle = \alpha\,|\lambda_1\rangle\,|\phi_1\rangle + \beta\,|\lambda_2\rangle\,|\phi_2\rangle\,, \tag{1}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\phi_i\rangle$ is Bob's corresponding state. The Born rule permits us to interpret $|\alpha|^2$ as the probability of Alice measuring λ_1, and $|\beta|^2 = 1 - |\alpha|^2$ as the probability she measures λ_2. Let's notate these as $p_1 = |\alpha|^2$ and $p_2 = |\beta|^2$.

Before we consider actually performing any measurement, let us first compute the reduced density matrix ρ_B of Bob's qubit via the partial trace of this arbitrary state. We trace out Alice's qubit, choosing her outcome states $|\lambda_i\rangle$ as the enumerated basis.

$$\rho_B = \mathrm{Tr}_A\Big(|\psi\rangle\langle\psi|\Big) = \sum_i \Big(\langle\lambda_i|\otimes\hat{\mathbb{1}}\Big)|\psi\rangle\langle\psi|\Big(|\lambda_i\rangle\otimes\hat{\mathbb{1}}\Big). \tag{2}$$

We substitute in our general pure state

$$\begin{aligned}|\psi\rangle\langle\psi| &= \Big(\alpha\,|\lambda_1\rangle\,|\phi_1\rangle + \beta\,|\lambda_2\rangle\,|\phi_2\rangle\Big)\Big(\alpha^*\,\langle\lambda_1|\,\langle\phi_1| + \beta^*\,\langle\lambda_2|\,\langle\phi_2|\Big) &(3)\\ &= |\alpha|^2\,|\lambda_1,\phi_1\rangle\langle\lambda_1,\phi_1| + \alpha\beta^*\,|\lambda_1,\phi_1\rangle\langle\lambda_2,\phi_2| + \beta\alpha^*\,|\lambda_2,\phi_2\rangle\langle\lambda_1,\phi_1| + |\beta|^2\,|\lambda_2,\phi_2\rangle\langle\lambda_2,\phi_2| &(4)\end{aligned}$$

although spare ourselves the nuisance of handling every projector by appreciating that $\langle\lambda_1|\lambda_2\rangle = 0$, so that the partial trace simplifies to

$$\begin{aligned}\rho_B &= |\alpha|^2\,|\phi_1\rangle\langle\phi_1| + |\beta|^2\,|\phi_2\rangle\langle\phi_2| &(5)\\ &= p_1\,|\phi_1\rangle\langle\phi_1| + p_2\,|\phi_2\rangle\langle\phi_2|\,. &(6)\end{aligned}$$

Let us now consider that Alice *does* perform her measurement. The shared state collapses to either

$$|\psi\rangle \to \begin{cases} |\lambda_1\rangle\,|\phi_1\rangle\,, & \text{with probability } p_1,\\ |\lambda_2\rangle\,|\phi_2\rangle\,, & \text{with probability } p_2.\end{cases} \tag{7}$$

The post-measurement system can be in one of multiple states as per the specified probabilities. We can describe such a state, encoding the classical randomness (i.e. the outcome state) using a density matrix ! The post-measurement mixed state is simply written down as :

$$\begin{aligned} \rho' &= p_1 \left(\rho_{A=\lambda_1}\right) + p_2 \left(\rho_{A=\lambda_2}\right) && (8) \\ &= p_1 \ |\lambda_1, \phi_1\rangle \langle\lambda_1, \phi_1| + p_2 \ |\lambda_2, \phi_2\rangle \langle\lambda_2, \phi_2| \,. && (9) \end{aligned}$$

Let us now again trace out Alice's qubit to obtain the reduced density matrix of Bob's qubit.

$$\begin{aligned} \rho'_B = \mathrm{Tr}_A(\rho) &= \sum_i \Big(\langle\lambda_i| \otimes \hat{\mathbb{1}} \Big) \rho \Big(|\lambda_i\rangle \otimes \hat{\mathbb{1}} \Big) && (10) \\ &= p_1 \, |\phi_1\rangle \langle\phi_1| + p_2 \, |\phi_2\rangle \langle\phi_2| \,. && (11) \end{aligned}$$

Lo and behold, this is precisely the expression we found for Bob's state when Alice did *not* perform a prior measurement. We have ergo proven that Alice's measurement has no affect on Bob's state, nor the statistics of his subsequent measurements. Alice *cannot* communicate to Bob via her measurement using two shared qubits, entangled or otherwise.

Any number of any-level systems

If we want to be really rigorous, we should generalise our proof to permit Alice and Bob to each have *any* dimension subspaces (e.g. many *qutrits*, or their own continuously parameterised systems !). For illustration, let's now do this using a different logic than used above, which will make use of some properties of projectors and traces you have not yet seen ! We permit Alice and Bob to each have one partition of any quantum state $|\psi\rangle$. We'll notate operators upon their respective partitions as $\hat{A} \otimes \hat{\mathbb{1}}$ and $\hat{\mathbb{1}} \otimes \hat{B}$ respectively.

Let $\{\hat{\Pi}_i\}$ be projectors corresponding to Alice's possible outcomes when performing some measurement on her partition. Given Alice is no longer measuring one qubit, there could be many more than two such projectors. The possible outcome states can be expressed in terms of their projectors as :

$$|\psi\rangle \to \Big\{ \, |\psi_i\rangle = \frac{1}{\sqrt{p_i}} \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \, |\psi\rangle \, : \, i \Big\}, \tag{12}$$

where we have renormalised the post-projector states via the probabilities of their corresponding measurement outcomes, $p_i = \langle\psi| \left(\hat{\Pi}_i \otimes \hat{\mathbb{1}}\right) |\psi\rangle$. The output state after Alice's measurement can be expressed as a single mixed state :

$$\rho = \sum_i p_i \, |\psi_i\rangle \langle\psi_i| = \sum_i p_i \left(\frac{1}{\sqrt{p_i}} \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \right) |\psi\rangle \langle\psi| \left(\frac{1}{\sqrt{p_i}} \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \right)^\dagger = \sum_i \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \, |\psi\rangle \langle\psi| \, \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big), \tag{13}$$

where we leveraged that projectors are self-adjoint, i.e. $\hat{\Pi}_i = \hat{\Pi}_i^\dagger$. The reduced density matrix of Bob's partition after Alice's measurement is

$$\rho_B = \mathrm{Tr}_A(\rho) = \mathrm{Tr}_A \left(\sum_i \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \, |\psi\rangle \langle\psi| \, \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \right). \tag{14}$$

Happily, we will do need even need to evaluate this partial trace ! We can instead simplify it using some of its properties, such as linearity :

$$\rho_B = \sum_i \mathrm{Tr}_A \left(\Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \, |\psi\rangle \langle\psi| \, \Big(\hat{\Pi}_i \otimes \hat{\mathbb{1}} \Big) \right) \tag{15}$$

We will next shuffle around some operators. Beware that unlike the trace which is cyclic, i.e. $\mathrm{Tr}\Big(\hat{L}_1 \hat{L}_2 \hat{L}_3\Big) = \mathrm{Tr}\Big(\hat{L}_3 \hat{L}_1 \hat{L}_2\Big) = \mathrm{Tr}\Big(\hat{L}_2 \hat{L}_3 \hat{L}_1\Big)$, the patrial trace is only cyclic with respect to operators of the traced subspace. This is easy to demonstrate :

$$\mathrm{Tr}_{\mathrm{left}}\left((\hat{L}_1\otimes\hat{R}_1)(\hat{L}_2\otimes\hat{R}_2)(\hat{L}_3\otimes\hat{R}_3)\right)=\mathrm{Tr}_{\mathrm{left}}\left((\hat{L}_1\hat{L}_2\hat{L}_3)\otimes(\hat{R}_1\hat{R}_2\hat{R}_3)\right) \tag{16}$$

$$=\mathrm{Tr}\left(\hat{L}_1\hat{L}_2\hat{L}_3\right)(\hat{R}_1\hat{R}_2\hat{R}_3)=\mathrm{Tr}\left(\hat{L}_3\hat{L}_1\hat{L}_2\right)(\hat{R}_1\hat{R}_2\hat{R}_3)=\mathrm{Tr}\left(\hat{L}_2\hat{L}_3\hat{L}_1\right)(\hat{R}_1\hat{R}_2\hat{R}_3) \tag{17}$$

$$=\mathrm{Tr}_{\mathrm{left}}\left((\hat{L}_3\otimes\hat{R}_1)(\hat{L}_1\otimes\hat{R}_2)(\hat{L}_2\otimes\hat{R}_3)\right)=\mathrm{Tr}_{\mathrm{left}}\left((\hat{L}_2\otimes\hat{R}_1)(\hat{L}_3\otimes\hat{R}_2)(\hat{L}_1\otimes\hat{R}_3)\right) \tag{18}$$

and holds true even when some operators are not separable (like $|\psi\rangle\langle\psi|$ in our case), which we could show by expressing them as a weighted sum in a separable basis and expanding via linearity. By cyclity, Bob's state becomes

$$\rho_B=\sum_i \mathrm{Tr}_A\left(\left((\hat{\Pi}_i\hat{\Pi}_i)\otimes\hat{\mathbb{1}}\right)|\psi\rangle\langle\psi|\left(\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}\right)\right) \tag{19}$$

$$=\sum_i \mathrm{Tr}_A\left(\left(\hat{\Pi}_i\otimes\hat{\mathbb{1}}\right)|\psi\rangle\langle\psi|\right), \tag{20}$$

via idempotency of $\hat{\Pi}_i$ (i.e. $\hat{\Pi}_i\hat{\Pi}_i=\hat{\Pi}_i$). Let's now move the sum around the only terms affected by it, utilising linearity of both the partial trace and the tensor product, to express

$$\rho_B=\mathrm{Tr}_A\left(\left(\left(\sum_i\hat{\Pi}_i\right)\otimes\hat{\mathbb{1}}\right)|\psi\rangle\langle\psi|\right). \tag{21}$$

Finally, we recognise that the sum of projectors of all orthonormal outcome states is the identity operator ;

$$\sum_i\hat{\Pi}_i=\mathbb{1} \tag{22}$$

To appreciate this, think about applying it to any particular state, when expressing that state in this perfectly valid basis. Our algebra above has concluded that the reduced density matrix describing Bob's qubit after Alice's measurement is :

$$\rho_B=\mathrm{Tr}_A\left(|\psi\rangle\langle\psi|\right). \tag{23}$$

We immediately recognise this is identical to Bob's reduced density matrix when Alice does *not* perform any prior measurement. Alice cannot signal to Bob via her measurement basis no matter *what* quantum state they share !

Problem 2 : Density operator, partial trace, information and measure

Alice and Bob share state

$$|\psi\rangle=\frac{|01\rangle-|10\rangle}{\sqrt{2}} \tag{24}$$

1. What is the density matrix of the system $\hat{\rho}$ with 2 qubits ?
2. Verify that it is a pure state by calculating $\mathrm{Tr}(\hat{\rho}^2)$.

Note $\hat{\rho}_B=\mathrm{Tr}_A\,\hat{\rho}$ the density matrix obtained by partial trace on Alice's qubit. This matrix is an operator on the Hilbert space of the second qubit (Bob's), and reflects the information available to Bob.

3. Calculate $\hat{\rho}_B$ and link that result to the probability Bob has to get outcome 0 or 1 when he measures his qubit in the computational basis (We will write $\hat{O}$ the corresponding observable). Also verify that we have $\langle\hat{O}\rangle=\mathrm{Tr}[\hat{\rho}(\mathbb{1}\otimes\hat{O})]=\mathrm{Tr}(\hat{\rho}_B\hat{O})$.
4. Does the matrix ρ_B describe a pure state of the second qubit ? Justify by calculating $\mathrm{Tr}(\hat{\rho}_B^2)$. What about if the 2 qubit state $|\psi\rangle$ being separable in the form $|\psi_A\rangle\otimes|\psi_B\rangle$? We sometimes say that statistical mixtures of the state of a system is the fruit of entanglement of this system with its environment ; how can we interpret this in the light of the previous results ?

We admit that when the measurement of an observable $\hat{M}$ on the system gives the result m, then the density matrix ($\hat{\rho}$ before measurement) reads

$$\hat{\rho}' = \frac{\hat{P}_m \hat{\rho} \hat{P}_m^\dagger}{\mathrm{Tr}(\hat{P}_m^\dagger \hat{P}_m \hat{\rho})}, \tag{25}$$

where $\hat{P}_m$ is the projector on the subspace relative to m.

5. What is the state with 2 qubits $|\psi'\rangle$ obtained when Alice measure her qubit in the computational basis and finds 0 ? Compare $|\psi'\rangle\langle\psi'|$ and $\hat{\rho}'$.
6. When Alice measures her qubit on state $|\psi\rangle$ and finds 0, what is the density matrix ? $\hat{\rho}'_B$? Comment.

Solution

1. The density matrix of the system is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}|01\rangle\langle 01| - \frac{1}{2}|01\rangle\langle 10| - \frac{1}{2}|10\rangle\langle 01| + \frac{1}{2}|10\rangle\langle 10|, \tag{26}$$

and its matrix form in the computational basis reads

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

2. We have $\hat{\rho}^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \hat{\rho}$, which we could also verify by squaring (26) and (27). This implies $\mathrm{Tr}(\hat{\rho}^2) = 1$ and characterises a pure state (which we knew by construction).
3. The reduced density matrix describing Bob's qubit is given by the partial trace, tracing over Alice's basis states.

$$\rho_B = \mathrm{Tr}_A(\rho) = \sum_{|\phi\rangle \in \{|0\rangle, |1\rangle\}} \left(\langle\phi| \otimes \hat{\mathbb{1}}\right) \rho \left(|\phi\rangle \otimes \hat{\mathbb{1}}\right) \tag{28}$$

You can expand and evaluate this in the usual approach, although it is perhaps clearer to explicitly invoke that the matrix element $(\rho_B)_{ij}$ of the reduced state ρ_B is given by

$$\langle i|\rho_B|j\rangle = \sum_x \langle x|\langle i|\rho|x\rangle|j\rangle. \tag{29}$$

We can compute each of the 4 matrix elements separately. Using Eq. (26), the first element is

$$\langle 0|\rho_B|0\rangle = \sum_x \langle x0|\rho|x0\rangle = \langle 00|\rho|00\rangle + \langle 10|\rho|10\rangle = 0 + \frac{1}{2} = \frac{1}{2}. \tag{30}$$

One can proceed similarly to get the other 3 matrix elements. The reduced density matrix of Bob in matrix form is then

$$\rho_B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \tag{31}$$

which corresponds to a statistical mixture of $|0\rangle$ and $|1\rangle$, with equal probability 1/2.
If $O \equiv |1\rangle\langle 1|$, we now want to show that $\langle O\rangle = \mathrm{Tr}[\rho(\mathbb{1} \otimes O)] = \mathrm{Tr}(\rho_B O)$. In matrix form, the observable O is given by

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{32}$$

Using matrix notation we find the result of interest

$$\mathrm{Tr}[\rho(\mathbb{1} \otimes O)] = \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}, \tag{33}$$

$$\mathrm{Tr}(\rho_B O) = \mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}. \tag{34}$$

4. The density matrix ρ_B describes a mixed state since $\text{Tr}(\rho_B^2) = \text{Tr}(\mathbb{1}/4) = 1/2 < 1$. If $|\psi\rangle$ is separable in the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, then the density matrix becomes

$$\rho = |\psi_A \psi_B\rangle \langle \psi_A \psi_B| = \rho_1 \otimes \rho_2, \tag{35}$$

with

$$\rho_1 = |\psi_A\rangle \langle \psi_A| , \tag{36}$$
$$\rho_2 = |\psi_B\rangle \langle \psi_B| , \tag{37}$$

and the density matrix of the second qubit, that is

$$\hat{\rho}_B = \text{Tr}_A(\hat{\rho}_1 \otimes \hat{\rho}_2) = (\text{Tr}\hat{\rho}_1)\hat{\rho}_2 = \hat{\rho}_2 = |\psi_B\rangle \langle \psi_B| , \tag{38}$$

is indeed a pure state. We have thus illustrated by an example, the fact that if the total system is a pure state, a subsystem appears mixed iff it is entangled with the rest of the system.

5. If Alice measures her qubit and finds 0, it means the 2 qubits state obtained is $|\psi'\rangle = |01\rangle$, in which case

$$|\psi'\rangle\langle\psi'| = |01\rangle\langle 01|. \tag{39}$$

Now to compute ρ', we need the projector on the subspace 0, that is,

$$P_0 = |0\rangle\langle 0| \otimes \mathbb{1} = P_0^\dagger. \tag{40}$$

The computation is then straightforward

$$P_0\rho = \frac{1}{2} |01\rangle \langle 01| - \frac{1}{2} |01\rangle \langle 10| , \tag{41}$$
$$\hat{P}_0\rho P_0^\dagger = \frac{1}{2} |01\rangle \langle 01| , \tag{42}$$
$$\text{Tr}(P_0^\dagger P_0 \rho) = \text{Tr}(P_0 \rho P_0^\dagger) = \frac{1}{2}, \tag{43}$$

and the new density operator is

$$\rho' = \frac{P_0 \rho P_0^\dagger}{\text{Tr}(P_0^\dagger P_0 \rho)} = |01\rangle \langle 01| = |\psi'\rangle \langle \psi'| , \tag{44}$$

6. Again, if Alice measures her qubit and finds 0, it means the 2 qubits state obtained is $|\psi'\rangle = |01\rangle = |0\rangle_A \otimes |1\rangle_B$. That is, it is separable (or, equivalently, it is a "product state"). As a result, according to the discussion above about the density matrix of separable states, we can directly write down Bob's reduced density matrix as $\rho'_B = |1\rangle\langle 1|$.

Problem 3 : Decoherence.

Consider a composite system that is prepared in the initial state $|\psi\rangle = \sum_j c_j |E_j\rangle_A \otimes |\phi\rangle_B$ and evolves under a Hamiltonian $H_{AB} = \sum_j |E_j\rangle\langle E_j|_A \otimes H_B^{(j)}$ for time t.

a) Find an expression for the reduced states $\rho_A(t)$ and $\rho_B(t)$ of systems A and B as a function of time.

The evolution of the full system is given by the time-independent unitary evolution operator :

$$|\psi(t)\rangle = e^{-it\hat{H}_{AB}} |\psi(0)\rangle = e^{-it\sum_j |E_j\rangle\langle E_j|_A \otimes \hat{H}_B^{(j)}} \sum_k c_k |E_k\rangle_A \otimes |\phi\rangle_B . \tag{45}$$

We could immediately recognise the spectral theorem, as described here, but let's instead Taylor expand the exponential, invoking

$$\hat{U}(t) = e^{-it\sum_j |E_j\rangle\langle E_j|_A \otimes \hat{H}_B^{(j)}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-it \sum_j |E_j\rangle \langle E_j|_A \otimes \hat{H}_B^{(j)} \right)^n \tag{46}$$

$$= \sum_{n=0}^{\infty} \frac{t^n(-i)^n}{n!} \sum_j |E_j\rangle \langle E_j|_A \otimes \left(\hat{H}_B^{(j)}\right)^n . \tag{47}$$

The final step above leveraged the orthogonality of $\{|E_j\rangle\langle E_j|\}$, i.e. that $\langle E_j|E_{j'}\rangle = \delta_{j,j'}$. To appreciate this, consider $n = 2$ where

$$\left(\sum_j |E_j\rangle\langle E_j|_A \otimes \hat{H}_B^{(j)}\right)^2 = \left(\sum_j |E_j\rangle\langle E_j|_A \otimes \hat{H}_B^{(j)}\right)\left(\sum_{j'} |E_{j'}\rangle\langle E_{j'}|_A \otimes \hat{H}_B^{(j')}\right) \tag{48}$$

$$= \sum_j \sum_{j'} \left(|E_j\rangle\langle E_j|_A \otimes \hat{H}_B^{(j)}\right)\left(|E_{j'}\rangle\langle E_{j'}|_A \otimes \hat{H}_B^{(j')}\right) \tag{49}$$

$$= \sum_j \sum_{j'} |E_j\rangle\langle E_j|E_{j'}\rangle\langle E_{j'}|_A \otimes \hat{H}_B^{(j)}\hat{H}_B^{(j')} \tag{50}$$

$$= \sum_j |E_j\rangle\langle E_j|_A \otimes (\hat{H}_B^{(j)})^2, \tag{51}$$

which is straightforward to generalise for higher n. We continue from Eq. 47, moving the projectors outside of the Taylor expansion, which we then restore to an exponential :

$$\hat{U}(t) = \sum_j |E_j\rangle\langle E_j|_A \otimes \sum_{n=0}^{\infty} \frac{t^n(-i)^n}{n!}\left(\hat{H}_B^{(j)}\right)^n = \sum_j |E_j\rangle\langle E_j|_A \otimes e^{-it\hat{H}_B^{(j)}} \tag{52}$$

Having simplified the unitary time evolution operator, we apply it upon the initial state of the full system to find the full state at time t.

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \left(\sum_j |E_j\rangle\langle E_j|_A \otimes e^{-it\hat{H}_B^{(j)}}\right)\sum_k c_k |E_k\rangle_A \otimes |\phi\rangle_B \tag{53}$$

$$= \sum_j c_j |E_j\rangle_A \otimes e^{-it\hat{H}_B^{(j)}}|\phi\rangle_B\,, \tag{54}$$

again using orthogonality of $\{E_j\}$. Expressed as a density matrix, this is

$$|\psi(t)\rangle\langle\psi(t)| = \left(\sum_j c_j |E_j\rangle_A \otimes e^{-it\hat{H}_B^{(j)}}|\phi\rangle_B\right)\left(\sum_{j'} c_{j'}^* \langle E_{j'}|_A \otimes \langle\phi|_B\, e^{it\hat{H}_B^{(j')\dagger}}\right) \tag{55}$$

$$= \sum_j \sum_{j'} c_j c_{j'}^* |E_j\rangle\langle E_{j'}|_A \otimes e^{-it\hat{H}_B^{(j)}}|\phi\rangle\langle\phi|_B\, e^{it\hat{H}_B^{(j')}} \tag{56}$$

There, we used that each $\{\hat{H}_B^{(j)}\}$ are Hermitian, since the Hamiltonian $\hat{H}_{AB}$ is Hermitian, as are the projectors $\{|E_j\rangle\langle E_j|\}$. Since these projectors are orthogonal, $\hat{H}_{AB} = \hat{H}_{AB}^\dagger \implies \hat{H}_B^{(j)} = \hat{H}_B^{(j)\dagger}\ \forall\, j$.

The reduced density matrices of each partition is then found via the partial trace. Because we have expressed the state as (a sum of) terms separable between the partitions, it is trivial to evaluate the partial trace as

$$\rho_B(t) = \mathrm{Tr}_A\left(|\psi(t)\rangle\langle\psi(t)|\right) = \sum_j \sum_{j'} c_j c_{j'}^*\, \mathrm{Tr}_A\left(|E_j\rangle\langle E_{j'}|_A \otimes e^{-it\hat{H}_B^{(j)}}|\phi\rangle\langle\phi|_B\, e^{it\hat{H}_B^{(j')\dagger}}\right) \tag{57}$$

$$= \sum_j \sum_{j'} c_j c_{j'}^*\, \mathrm{Tr}\left(|E_j\rangle\langle E_{j'}|_A\right) e^{-it\hat{H}_B^{(j)}}|\phi\rangle\langle\phi|_B\, e^{it\hat{H}_B^{(j')}} \tag{58}$$

and finally, since $\mathrm{Tr}\left(|E_j\rangle\langle E_{j'}|_A\right) = \delta_{j,j'}$ (which we can intuit through thinking of the matrix form of $|E_j\rangle\langle E_{j'}|$), we conclude

$$\rho_B(t) = \sum_j |c_j|^2 e^{-it\hat{H}_B^{(j)}}|\phi\rangle\langle\phi|_B\, e^{it\hat{H}_B^{(j)}}. \tag{59}$$

We observe this is a mixture (with probabilities $|c_j|^2$) of the pure states resulting from unitary-time evolution under a random choice of Hamiltonian $\hat{H}_B^{(j)}$ upon state $|\phi\rangle_B$. Meanwhile,

$$\rho_A(t) = \mathrm{Tr}_B\Big(|\psi(t)\rangle\langle\psi(t)|\Big) = \sum_j\sum_{j'} c_j c_{j'}^* |E_j\rangle\langle E_{j'}|_A \,\mathrm{Tr}\Big(e^{-it\hat{H}_B^{(j)}}|\phi\rangle\langle\phi|_B\, e^{it\hat{H}_B^{(j')}}\Big) \tag{60}$$

$$= \sum_j\sum_{j'} c_j c_{j'}^* |E_j\rangle\langle E_{j'}|_A \langle\phi|\, e^{it\hat{H}_B^{(j')}} e^{-it\hat{H}_B^{(j)}}|\phi\rangle . \tag{61}$$

The final step used the cyclic property of the trace, and that the trace of a scalar is the scalar :

$$\mathrm{Tr}\Big(\hat{C}|\phi\rangle\langle\phi|\hat{D}\Big) = \mathrm{Tr}\Big(|\phi\rangle\langle\phi|\hat{D}\hat{C}\Big) = \mathrm{Tr}\Big(\langle\phi|\hat{D}\hat{C}|\phi\rangle\Big) = \langle\phi|\hat{D}\hat{C}|\phi\rangle$$

Beware that because $e^{it\hat{H}_B^{(j')}}$ and $e^{-it\hat{H}_B^{(j)}}$ do not necessarily commute, we cannot express them as a single exponential.

b) Under what circumstances do A and B remain pure for all times ?

Given the full system begins and remains in a pure state, the reduced density matrices of each partition remain pure whenever the full state remains separable. That is, only when the partitions become entangled does each partition's state becomes mixed. There are several methods to find constraints on the Hamiltonian which admit this circumstance, such as explicitly asserting $\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2) = 1$, observing when ρ_A or ρ_B have the form of a pure state $|\Psi\rangle\langle\Psi|$, or observing when the full system Hamiltonian $\hat{H}_{AB}$ upon the given initial state $|\psi\rangle$ never induces entanglement.

All three methods constrain that $\hat{H}_B^{(j)}$ are fixed across j, such that

$$\hat{H}_{AB} = \left(\sum_j |E_j\rangle\langle E_j|\right)\otimes \hat{H}_B. \tag{62}$$

Erratum

A previous solution erroneously asserted that entanglement is never generated when the full system Hamiltonian is *seperable*, i.e. when $\hat{H}_{AB} = \hat{H}_A \otimes \hat{H}_B$, regardless of the state which is time-evolved. This is not true in general ; the evolution under even separable Hamiltonians can generate entanglement. For example, consider when $\hat{H}_A = \hat{H}_B = \hat{X}$, whereby the full evolution of the zero state

$$e^{-it\hat{H}_{AB}}|00\rangle = e^{-it\hat{X}_A\otimes\hat{X}_B}|00\rangle = \cos(t)|00\rangle - i\sin(t)|11\rangle ,$$

produces entangled states for all $t = n\pi/2$, for any integer $n \in \mathbb{N}$. In this problem, it is not necessary for the Hamiltonian $\hat{H}_{AB}$ to be incapable of generating entanglement when evolving *any* initial state ; we care only about the evolution of the given state $|\psi\rangle$.

For a two-partite Hamiltonian to never generate entanglement, it must be expressible in the form

$$\hat{H}_{AB} = \hat{H}_A \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{H}_B,$$

where commutation of the terms simplifies the unitary-time evolution operator to

$$\begin{aligned} e^{-it\hat{H}_{AB}} &= e^{-it\hat{H}_A\otimes\hat{\mathbb{1}}}\, e^{-it\hat{\mathbb{1}}\otimes\hat{H}_B} \\ &= \Big(e^{-it\hat{H}_A}\otimes\hat{\mathbb{1}}\Big)\Big(\hat{\mathbb{1}}\otimes e^{-it\hat{H}_B}\Big) && \text{(evident by Taylor expansion)} \\ &= e^{-it\hat{H}_A}\otimes e^{-it\hat{H}_B}, \end{aligned}$$

which as a separable *unitary*, generates no entanglement.

c) Under what circumstances does $\rho_A(t)$ become approximately diagonal in the basis $\{|E_j\rangle\}$?

We found

$$\rho_A(t) = \sum_i\sum_j \Big(c_i c_j^* \;\langle\phi|\, e^{it\hat{H}_B^{(j)}} e^{-it\hat{H}_B^{(i)}}|\phi\rangle\Big)|E_i\rangle\langle E_j|_A , \tag{63}$$

where the coefficient of $|E_i\rangle\langle E_j|_A$ is the (i,j)-th element of the density matrix in the basis of $\{|E_j\rangle\}$. This is approximately diagonal whenever the off-diagonals (where $i \neq j$) are approximately zero. Ergo

$$c_i c_j^* \langle\phi| e^{it\hat{H}_B^{(j)}} e^{-it\hat{H}_B^{(i)}} |\phi\rangle \approx 0 \quad \forall\, i \neq j. \tag{64}$$

Rejecting $c_i \approx 0\ \forall\ i$ which invalidates our initial state, we recognise this as the constraint that state $e^{-it\hat{H}_B^{(i)}}|\phi\rangle$ is approximately orthogonal to state $e^{-it\hat{H}_B^{(j)}}|\phi\rangle$. This means that the evolution to time t under each local Hamiltonian on partition B drives the partition to almost orthogonal states. We can interpret this as the B partition acting like a measuring device upon the A partition, driving it into a pointer state $|E_j\rangle$.

Problem 4 : Decoherence and dephasing of a single qubit

1. Consider applying a random R_z rotation, i.e. $e^{-i\vartheta/2\sigma_z}$ for $\vartheta/2 \in [-\pi,\pi]$ to a generic initial pure qubit state $|\psi\rangle = \sin(\theta)|0\rangle + \cos(\theta)e^{-i\phi}|1\rangle$. What is the resulting mixed state on average? Sketch this on the Bloch sphere.

   Disclaimer : What does "average mixed state" even mean? In this case, it means that we take the average over all states with respect to this σ_z rotation. How do we take the average? We just compute
$$\rho_{\text{av}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_z(\vartheta)\,|\psi\rangle\langle\psi|\,R_z^\dagger(\vartheta)d\vartheta \tag{65}$$

   The initial density matrix is
$$\rho_0 = |\psi\rangle\langle\psi| = \begin{pmatrix} \sin^2(\theta) & e^{i\phi}\sin(\theta)\cos(\theta) \\ e^{-i\phi}\sin(\theta)\cos(\theta) & \cos^2(\theta)\end{pmatrix} \tag{66}$$

   Now we can apply a random rotation R_z on the generic initial state. To do so, we expand $R_z(\vartheta) = \cos\frac{\vartheta}{2}\mathbb{1} - i\sin\frac{\vartheta}{2}Z$. We now average over $\vartheta \in [-\pi,\pi]$ by taking the integral.
$$\langle\rho_z(\vartheta)\rangle_\vartheta = \langle\cos^2\frac{\vartheta}{2}\rangle_\vartheta\rho_0 + i\langle\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}\rangle_\vartheta(Z\rho_0 - \rho_0 Z) + \langle\sin^2\frac{\vartheta}{2}\rangle_\vartheta Z\rho_0 Z \tag{67}$$
$$= \frac{1}{2}(\rho_0 + Z\rho_0 Z) = \begin{pmatrix}\sin^2\theta & 0 \\ 0 & \cos^2\theta\end{pmatrix} \tag{68}$$

   And if we want to see the effect of this rotation on the Block sphere, it just rotates the state around the z axis by an angle ϑ. If we compute the average it would be a vector with an angle $\phi = 0$.

2. Consider now applying a random R_z rotation and then a random R_x rotation. What is the resulting mixed state on average? And what if you now apply all three (a random R_z, R_x and R_y)? Sketch this on the Bloch sphere.

   In the previous point, we observed that only the terms where the Pauli matrices act symmetrically on both sides of the density matrix survive the averaging, as $\langle\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\rangle = 0$. Using this, and the fact that the angles for the two rotations are independently sampled, i.e. $\langle\cdot\rangle_{\vartheta,\gamma} = \langle\cdot\rangle_\vartheta\langle\cdot\rangle_\gamma$, we find
$$\langle\rho_{x,z}(\gamma,\vartheta)\rangle_\vartheta = \langle\cos^2\frac{\vartheta}{2}\rangle_\vartheta\langle\cos^2\frac{\gamma}{2}\rangle_\gamma\rho_0 + \langle\cos^2\frac{\vartheta}{2}\rangle_\vartheta\langle\sin^2\frac{\gamma}{2}\rangle_\gamma X\rho_0 X \tag{69}$$
$$+ \langle\sin^2\frac{\vartheta}{2}\rangle_\vartheta\langle\cos^2\frac{\gamma}{2}\rangle_\gamma Z\rho_0 Z + \langle\sin^2\frac{\vartheta}{2}\rangle_\vartheta\langle\sin^2\frac{\gamma}{2}\rangle_\gamma XZ\rho_0 ZX \tag{70}$$
$$= \frac{1}{4}(\rho_0 + X\rho_0 X + Z\rho_0 Z + XZ\rho_0 ZX) = \frac{1}{4}(\rho_0 + X\rho_0 X + Z\rho_0 Z + Y\rho_0 Y) \tag{71}$$
$$= \begin{pmatrix}\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{pmatrix}. \tag{72}$$

   We already obtain the maximally mixed state after averaging over X and Z rotations, a third rotation about the Y axis can hence not mix the state further. For completeness, we can still compute

$$\rho_{zxy}(\vartheta,\gamma,\beta) = R_y(\beta)\rho_{zx}R_y^\dagger(\beta) \tag{73}$$

$$= \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \rho_{zx} \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \tag{74}$$

Now we can compute the average of the density matrix.

$$\langle \rho_{zxy}(\vartheta,\gamma,\beta)\rangle_{\vartheta,\gamma,\beta} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{75}$$

3. What if instead you apply a random R_z rotation with probability p and do nothing with probability $1-p$? And what if you apply a random R_z rotation then a random R_x rotation with probability p, and do nothing with probability $1-p$? Sketch this on the Bloch sphere.
   If we apply a random R_z rotation with probability p and do nothing with probability $1-p$, we can write the final density matrix as follows.

$$\rho_{z,\mathbb{I}} = pR_z(\vartheta)\rho R_z^\dagger(\vartheta) + (1-p)\rho \tag{76}$$

   And if we apply this on a general initial density matrix, we will have,

$$\rho_{z,\mathbb{I}} = p\begin{pmatrix} \sin^2(\theta) & e^{i(\phi-\vartheta)}\sin(\theta)\cos(\theta) \\ e^{-i(\phi-\vartheta)}\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} + (1-p)\begin{pmatrix} \sin^2(\theta) & e^{i\phi}\sin(\theta)\cos(\theta) \\ e^{-i\phi}\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \tag{77}$$

   And then if we compute the average over ϑ we will have,

$$\rho_{z,\mathbb{I}} = \begin{pmatrix} \sin^2(\theta) & (1-p)e^{i\phi}\sin(\theta)\cos(\theta) \\ (1-p)e^{-i\phi}\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \tag{78}$$

   If we apply a random R_z rotation and then apply a random R_x with probability p and do nothing with probability $1-p$, we can write the final density matrix as follows.

$$\rho_{zx,\mathbb{I}} = pR_x(\vartheta)\rho_z R_x^\dagger(\vartheta) + (1-p)\rho \tag{79}$$

   And if we apply this on the ρ_z and then if we compute the average over ϑ we will have,

$$\rho_{zx,\mathbb{I}} = \begin{pmatrix} (1-p)\sin^2(\theta) + \frac{p}{2} & (1-p)e^{i\phi}\sin(\theta)\cos(\theta) \\ (1-p)e^{-i\phi}\sin(\theta)\cos(\theta) & (1-p)\cos^2(\theta) + \frac{p}{2} \end{pmatrix} \tag{80}$$

4. Suppose you now instead throw away your initial state and prepare the maximally mixed state $\mathbb{I}/2$ with probability p and do nothing with probability $1-p$? Sketch this on the Bloch sphere.

$$\rho_{\mathbb{I}} = p\frac{\mathbb{I}}{2} + (1-p)\rho \tag{81}$$

$$\rho_{\mathbb{I}} = \begin{pmatrix} (1-p)\sin^2(\theta) + \frac{p}{2} & (1-p)e^{i\phi}\sin(\theta)\cos(\theta) \\ (1-p)e^{-i\phi}\sin(\theta)\cos(\theta) & (1-p)\cos^2(\theta) + \frac{p}{2} \end{pmatrix} \tag{82}$$

5. What do you conclude from all this?
   When we apply just one random rotation R_z with probability p and do nothing with probability $1-p$, it will be a dephasing quantum channel. When we apply random rotations R_x and R_z with probability p and do nothing with probability $1-p$ it will be a depolarizing quantum channel. And when we throw away the initial state with probability p and do nothing with probability $1-p$ it will be the depolarizing channel.